GEOMETRY OF CRITICAL LOCI

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0. *Introduction*

Let

$$\pi : (Z, z) \longrightarrow (U, 0)$$

be the germ of a finite (that is, proper with finite fibres) complex analytic morphism from a complex analytic normal surface onto an open neighbourhood U of the origin 0 in the complex plane $\mathbf{C}^2$. Let u and v be coordinates of $\mathbf{C}^2$ defined on U. We shall call the triple (π, u, v) the *initial data*.

Let Δ stand for the discriminant locus of the germ π, that is, the image by π of the critical locus Γ of π.

Let $(\Delta_\alpha)_{\alpha \in A}$ be the branches of the discriminant locus Δ at O which are not the coordinate axes.

For each $\alpha \in A$, we define a rational number d_α by

$$d_\alpha = \frac{I(u=0, \Delta_\alpha)}{I(v=0, \Delta_\alpha)}$$

where $I(-,-)$ denotes the intersection number at 0 of complex analytic curves in $\mathbf{C}^2$. The set of rational numbers d_α, for $\alpha \in A$, is a finite subset D of the set of rational numbers $\mathbf{Q}$. We shall call D the set of *discriminantal ratios* of the initial data (π, u, v). The interesting situation is when one of the two coordinates (u, v) is tangent to some branch of Δ, otherwise $D = \{1\}$. The definition of D depends not only on the choice of the two coordinates, but also on their ordering.

In this paper we prove that the set D is a topological invariant of the initial data (π, u, v) (in a sense that we shall define below) and we give several ways to compute it. These results are first steps in the understanding of the geometry of the discriminant locus. We shall also see the relation with the geometry of the critical locus.

In order to state our results, we give some definitions and fix the notations. Let M denote the boundary of a subanalytic neighbourhood W of z in Z which is adapted to $f = 0$ and $g = 0$, where $f := u \circ \pi$ and $g := v \circ \pi$ (see §2.1 below).

Let $K := f^{-1}(0) \cap M$ and $L := g^{-1}(0) \cap M$.

If W is sufficiently small, the topology of the link $K \cup L$ in M does not depend on the choice of W. By definition K and L are not empty and $K \cap L = \varnothing$.

The space M is a smooth differentiable manifold of dimension 3 which is compact, connected (because the point $z \in Z$ is a normal singularity of Z) and oriented as boundary of the complex manifold of non-singular points of W naturally endowed

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with the orientation due to the complex structure. The manifolds K and L are oriented in the same manner, since they are respectively boundaries of the manifold of smooth points of $f^{-1}(0) \cap W$ and $g^{-1}(0) \cap W$.

We call the triple (M, L, K) the *link* associated to the initial data (π, u, v).

Our arguments are based on the theory of 3-manifolds which are Waldhausen, as developed in [**21**, **25**]. In fact, there is a Waldhausen decomposition on M such that K and L are a union of Seifert leaves.

DEFINITION 0.1. The link (M, L, K) is *non-reduced* if M is a lens space and if $K \cup L$ are the cores of two solid tori whose union is M. In all the other cases, the link is said to be reduced.

We shall use the following theorem which is due to F. Waldhausen and W. Neumann (see Theorem 2.13 below).

THEOREM 0.2 (unicity of the minimal decomposition). *Let (M, K, L) be a reduced link. The minimal Waldhausen decomposition of M in which K and L are union of Seifert leaves is unique up to isotopy and its Seifert components are endowed with a Seifert foliation unique up to isotopy.*

We shall show that the link (M, L, K) associated to the initial data (π, u, v) is reduced when the set $(\Delta_\alpha)_{\alpha \in A}$ of branches of Δ at O which are not the coordinate axes is non-empty (see Remark 3.6).

From now on, we assume that there is at least one branch of Δ which is not a coordinate axis.

Let $(V_j)_{j \in J}$ be the set of Seifert components of the minimal decomposition of (M, K, L). For each index $j \in J$, let us choose a regular leaf γ_j. We define the ratio

$$t_j := \frac{\deg(f \mid \gamma_j)}{\deg(g \mid \gamma_j)}$$

where $\deg(f \mid \gamma_j)$ denotes the degree of the composition of the map induced by the restriction f to γ_j, which maps γ_j into $\mathbf{C}^*$, and the map from $\mathbf{C}^*$ into the unit circle $\mathbf{S}^1$ which sends a complex number t onto $t/|t|$.

We call the set of rational numbers $(t_j)_{j \in J}$ the set T of *topological quotients* of (π, u, v).

The main result of this paper is Theorem 0.3, which is proved in §3.

THEOREM 0.3. *We have $D = T$.*

The inclusion $T \subset D$ is a consequence of the fact that the minimal decomposition of (M, K, L) is obtained from the minimal decomposition of the complement of the discriminant Δ in a sphere of $\mathbf{C}^2$ centred at 0 with a sufficiently small radius. The opposite inclusion requires a more precise analysis of the relation between the two preceding minimal decompositions.

Now, let (M, K, L) and (M', K', L') be two links. We shall say that they are isomorphic if there is an orientation-preserving diffeomorphism $\Phi : M \longrightarrow M'$, such that $\Phi(K) = K'$, $\Phi(L) = L'$ and if Φ also preserves the orientations of K, K' and L, L'.

DEFINITION 0.4. Two initial data (π, u, v) and (π', u', v') are *topologically equivalent* if the associated links are isomorphic.

THEOREM 0.5. *The set T is a topological invariant of the initial data, that is, it only depends on the isomorphism class of the link associated to (π, u, v).*

COROLLARY 0.6. *The set D is a topological invariant of the initial data.*

We conclude our paper with some explicit examples.

1. *A special case, proper morphisms and variants of T and D*

1.1.

When z is non-singular in Z, the first interesting case is when f is any holomorphic function and g is a local coordinate of Z at z which is transverse to f.

In this case the set D is the set of polar ratios of f at z (see [**14**, (4.7); **24**, (3.5.1)]).

When f is analytically irreducible at z, the polar ratios of f at z have been computed in [**18**] by M. Merle in terms of Puiseux pairs of f at z, which showed the topological invariance of D in this case. When f is not analytically irreducible at z, the topological invariance was proved in [**15**].

From the viewpoint developed in this paper, the topological invariance in this former situation follows from the fact that, if two analytic functions f and g have the same topological type at z, then, for any local coordinate ξ of Z at z transverse to both f and g at z, the functions $f\xi$ and $g\xi$ have the same topological type at z.

When f and g are analytic functions, H. Maugendre has shown in [**17**] that the discriminantal ratios depend only on the topology of the function fg at z. In this case, since Z is smooth at z, the space M is a 3-sphere. Therefore, the degrees which appear in the definition of the topological ratios can be replaced by well-chosen linking numbers. This remark enables the reader to relate our results and those of H. Maugendre.

1.2.

There is an application of our theorems to the following situation. Consider a complex analytic map

$$\sigma : X \longrightarrow U,$$

from a normal complex surface X onto an open neighbourhood U of 0 in the complex plane $\mathbf{C}^2$, which is proper and such that the fibre $\sigma^{-1}(0)$ is a connected curve.

By choosing U small enough, we may assume that X is connected and that the restriction of σ from $X - \{\sigma^{-1}(O)\}$ onto $U - \{0\}$ is finite.

One can contract $\sigma^{-1}(0)$. By using the Stein factorization theorem, one obtains a normal singularity (Z, z) and a finite morphism

$$\pi : (Z, z) \longrightarrow (U, 0)$$

as in the introduction.

Instead of the Stein theorem, one can also use the Grauert contractibility criterion, because the curve $\sigma^{-1}(0)$ is the support of the compact part of the divisor of a holomorphic function on X which lifts a function of U which vanishes at 0.

1.3.

Variants of T and D are obtained as follows.

Let $\rho: Y \longrightarrow Z$ be the minimal resolution of the singularity (Z, z) in which the divisor $fg \circ \rho$ has normal crossings and the irreducible components of the exceptional divisor are non-singular (good resolution). That resolution can be obtained by looking at the resolutions (obtained by blowing up the minimal good resolution of Z) in which the divisor of fg is a normal crossing divisor. One then takes the minimal good resolution among these.

Definition 1.1. A component of the exceptional divisor of ρ is called a *rupture component* if it is not a rational curve or if it has at least three intersection points with the other components (compact or not) of the divisor of $fg \circ \rho$.

Let $(E_\beta)_{\beta \in B}$ be the set of irreducible components of the exceptional divisor which are rupture components. For each $\beta \in B$, we define

$$h_\beta := \frac{\mathrm{val}_{E_\beta}(f)}{\mathrm{val}_{E_\beta}(g)}$$

where val_{E_β} is the divisorial valuation defined by E_β on the local ring $\mathcal{O}_{Z,z}$.

The rational number h_β is called the *Hironaka number* of (f, g) on E_β and the set $(h_\beta)_{\beta \in B}$ is written H. We have the following proposition.

Proposition 1.2. $T = H$.

We shall give a proof of this result in the next section.

On the other hand, we can give the following interpretation of D.

Let $(\Gamma_\lambda)_{\lambda \in \Lambda}$ be the irreducible components of the critical locus of π which project onto components of Δ. For each $\lambda \in \Lambda$, we define

$$c_\lambda := \frac{I_z(f = 0, \Gamma_\lambda)}{I_z(g = 0, \Gamma_\lambda)}$$

where I_z is the intersection number on Z at z. We call c_λ the critical ratio associated to Γ_λ. We write C for the set of critical ratios. We have the following proposition.

Proposition 1.3. $D = C$.

This proposition will also be proved in the next section.

2. *The topology of the link*, *the topological invariance of T and the equalities $T = H$ and $D = C$*

2.1. *Subanalytic neighbourhoods*

First, we need to describe the setting in which we shall be able to deal with the topology associated to the initial data. We essentially follow the presentation of A. Durfee in [**2**].

Definition 2.1. Let E be a subanalytic set in $\mathbf{R}^N$ and let $F \subset E$ be a compact subanalytic subset. A *subanalytic rug function* for F in E is a proper subanalytic function $\varphi: E \longrightarrow \mathbf{R}$ such that $\varphi(x) \geqslant 0$ for $x \in E$ and $\varphi^{-1}(0) = F$.

Arguments similar to the ones of [**2**, Lemma 3.2] show the existence of subanalytic functions.

DEFINITION 2.2. Let E be a subanalytic set in $\mathbf{R}^N$ and let $F \subset E$ be a compact subanalytic subset. Let φ be a subanalytic rug function for F in E. For any $\epsilon > 0$, it defines neighbourhoods $U_\epsilon(\varphi) := \varphi^{-1}([0, \epsilon])$ of F in E.

When ϵ is small enough, the following theorem gives the unicity of such neighbourhoods.

THEOREM 2.3. *Let φ_1 and φ_2 be two subanalytic rug functions for F in E. Let $E_1, \dots, E_k$ be a finite family of subanalytic subsets of E which contain F. There exist $\epsilon(\varphi_1)$ and $\epsilon(\varphi_2)$ such that, for any ϵ_1, $0 < \epsilon_1 \leqslant \epsilon(\varphi_1)$ and any ϵ_2, $0 < \epsilon_2 \leqslant \epsilon(\varphi_2)$, there is a continuous family of stratified embeddings h_t, $0 \leqslant t \leqslant 1$, of $U_{\epsilon_1}(\varphi_1)$ into E such that*

(i) *h_0 is the inclusion;*
(ii) *h_t restricted to F is the inclusion, for any t;*
(iii) *for any t, h_t induces a family of stratified embeddings of $U_{\epsilon_1}(\varphi_1) \cap E_i$ into E_i, for any i, $1 \leqslant i \leqslant k$;*
(iv) *h_1 induces a stratified homeomorphism of $U_{\epsilon_1}(\varphi_1)$ onto $U_{\epsilon_2}(\varphi_2)$.*

The proof of this theorem is essentially the same as the one of [**2**, Proposition 3.5], considering the subanalytic category and stratifications of E adapted to the E_i, $1 \leqslant i \leqslant k$.

REMARK 2.4. If one considers $\varphi = \varphi_1 = \varphi_2$, Theorem 2.3 shows that, for any ϵ with $0 < \epsilon \leqslant \epsilon(\varphi)$, all the neighbourhoods U_ϵ are homeomorphic. We shall call such neighbourhoods of F in E *subanalytic neighbourhoods* of F in E.

In fact, a subanalytic neighbourhood of F in E is a regular neighbourhood of F in E. In particular, subanalytic neighbourhoods of F in E provide a system of good neighbourhoods of F in E with respect to the E_i, $1 \leqslant i \leqslant k$ (in the sense of Prill [**23**]).

When we apply these results to the situation that we consider in this paper, we have the following corollary.

COROLLARY 2.5. *The topology of the link (M, L, K) does not depend on the chosen subanalytic neighbourhood of $\{z\}$ in Z.*

Let $\rho\colon Y \longrightarrow Z$ be a resolution of the singularity (Z, z). Then, ρ is proper analytic and induces an isomorphism of $Y - \rho^{-1}(z)$ onto $Z - \{z\}$. Since (Z, z) is normal, if z is a genuine singular point, the fibre $\rho^{-1}(z)$ is a curve. Let E_j, $j \in J$, be the components of the curve $\rho^{-1}(z)$. For each component E_j, we can define subanalytic neighbourhoods $U^{(j)}_{\epsilon_j}(\varphi_j)$ of E_j in Y. We have the following lemma.

LEMMA 2.6. *There are subanalytic functions $\epsilon_j\colon [0, 1] \longrightarrow \mathbf{R}^+$, such that there is t_0, $0 < t_0 \leqslant 1$, for which*

$$\bigcup_{j \in J} U^{(j)}_{\epsilon_j(t)}(\varphi_j)$$

is a subanalytic neighbourhood of $\bigcup_{j \in J} E_j$ in Y, for any $t \leqslant t_0$.

2.2. *Relative Waldhausen decompositions*

We consider compact, connected, three-dimensional manifolds. We suppose that each of them is endowed with a fixed orientation. Such a manifold V is called a *Seifert fibred manifold* if it is equipped with a foliation by circles. We shall always suppose that these foliations are orientable. In this case a theorem of D. Epstein (see [**4**]) implies that the foliation is given by the orbits of a fixed-point free action of S^1 on V. As V is compact, there are a finite number of exceptional leaves which correspond to orbits with a finite non-trivial isotropy subgroup. The other leaves are called regular and have a trivial isotropy subgroup. For each exceptional leaf one can find foliated neighbourhoods which are solid tori. The action of S^1 on such a neighbourhood is described by a couple (α, β) where α, $\alpha \geqslant 2$, is the order of the isotropy subgroup and β is some integer only defined modulo α (see [**22**]).

The boundary components of a Seifert fibred manifold are tori (because they are oriented and foliated by circles).

REMARK 2.7. The fact that the foliations are assumed to be orientable implies that the orbit spaces are orientable surfaces.

DEFINITION 2.8. A *Waldhausen decomposition* of a compact connected oriented three-dimensional manifold W without boundary is a decomposition of W as $(V_j)_{j \in J}$, where J is finite, such that the following hold:

(i) Each V_j is a connected Seifert fibred manifold.

(ii) If $j \neq k$, the intersection $V_j \cap V_k$ is either empty or a finite union of tori which are contained in the boundary of V_j and of V_k. Such a torus is equipped with two foliations: one induced by V_j and the other by V_k.

If S is a link in W, that is, S is a disjoint union of circles contained in W, we have the following definition.

DEFINITION 2.9. A *Waldhausen decomposition of the pair* (W, S) is a Waldhausen decomposition of W such that each component of S is contained in the interior of some V_j and is a leaf of the Seifert structure on V_j. In this case we shall say that the Seifert structure of V_j is *compatible with the link* S.

From now on, we suppose that the link S is non-empty.

DEFINITION 2.10. A Seifert component V of a Waldhausen decomposition of (W, S) is called *non-intrinsic* if one of the following cases occurs:

(i) The boundary of V is empty (that is $V = W$), V is a lens-space and S is contained in the union of the cores of a decomposition of V in two solid tori.

(ii) The boundary of V is non-empty, $V \cap S$ is empty and V is a solid torus or a thickened torus.

(iii) The boundary of V is non-empty, $V \cap S$ is non-empty and V is a solid torus whose core is $V \cap S$.

In all other cases, a Seifert component is called *intrinsic*.

THEOREM 2.11. *On an intrinsic component there is, up to isotopy, a unique Seifert foliation compatible with* $V \cap S$. (*See* [**10**, p. 97].)

Definition 2.12. A Waldhausen decomposition $(V_j)_{j\in J}$ of (W, S) is *minimal* if the cardinality of J is minimal among all Waldhausen decompositions.

Results of F. Waldhausen [**25**], and W. Jaco and P. Shalen [**11**, **12**] adapted to our situation imply Theorem 0.2 of the introduction.

Theorem 2.13 (unicity of the minimal decomposition). *Let (M, K, L) be a reduced link. The minimal Waldhausen decomposition of M in which K and L are a union of Seifert leaves is unique up to isotopy and its Seifert components are endowed with a Seifert foliation unique up to isotopy. Moreover, let $\mathcal{U}$ be a minimal Waldhausen decomposition of $(M, K \cup L)$ and $\mathcal{V}$ be a Waldhausen decomposition of $(M, K \cup L)$, such that $\mathcal{U} \preccurlyeq \mathcal{V}$. Then, for each $U_i \in \mathcal{U}$, there exists a $V_j \in \mathcal{V}$, such that, up to isotopy, $V_j \subset U_i$ and each leaf of V_j is a leaf of U_i.*

As an immediate consequence, we have the following corollary.

Corollary 2.14. *Once the initial data is given, the set T of topological quotients is well-defined.*

2.3. *Plumbing calculus*

A special case of Waldhausen decompositions (in general not minimal) is obtained from plumbing. In such a construction the building blocks are differentiable complex line bundles over compact connected orientable differentiable surfaces without boundary. The total space is supposed to be oriented. Up to differentiable isomorphism, these bundles are classified by the genus of the base space and the self-intersection of the zero-section.

Given two bundles, one can plumb them at various points according to a process which is explained with precision in [**9**, pp. 66–67]. By applying this process a finite number of times, one obtains an oriented differentiable 4-manifold with corners. The plumbing instructions are encoded in a graph G called the *plumbing graph*, which describes the diffeomorphism class of the 4-manifold.

Let us go back to the end of §2.1. We use the same notations. If the resolution ρ is good as in §1.3, the subanalytic functions ϵ_j can be chosen such that

(i) the subanalytic neighbourhood $U^{(j)}_{\epsilon_j(t)}(\varphi_j)$ of E_j in Y is a tubular neighbourhood of E_j in Y, for any t small enough and any $j \in J$;

(ii) for $j \neq l$, the neighbourhoods $U^{(j)}_{\epsilon_j(t)}(\varphi_j)$ and $U^{(l)}_{\epsilon_l(t)}(\varphi_l)$ intersect in a finite disjoint union of polydiscs, for any t small enough;

(iii) the neighbourhood $\bigcup_{j\in J} U^{(j)}_{\epsilon_j(t)}(\varphi_j)$ is the plumbing of the neighbourhoods $U^{(j)}_{\epsilon_j(t)}(\varphi_j)$, $j \in J$, for t small enough.

As a consequence, the boundary ∂N of the neighbourhood $N = \bigcup_{j\in J} U^{(j)}_{\epsilon_j(t)}(\varphi_j)$ is a graph manifold in the sense of Waldhausen (Waldhausen manifold). An important consequence of Theorem 2.3 is the following theorem.

Theorem 2.15. *The* 3*-manifold M is a Waldhausen manifold and there is a Waldhausen decomposition in which L and K are Seifert leaves.*

Proof. We consider a good resolution ρ of Z in which the divisor of $fg \circ \rho$ is a normal crossing divisor. The inverse image by ρ of a subanalytic neighbourhood of

z in Z is a subanalytic neighbourhood of the fibre $\rho^{-1}(z)$ in Y. Theorem 2.3 applied to the case where $E = Y$, $F = \rho^{-1}(z)$, E_1 is the inverse image of $f = 0$ by ρ and E_2 is the inverse image of $g = 0$ by ρ shows that M is homeomorphic to the boundary ∂N of $N = \bigcup_{j\in J} U^{(j)}_{\epsilon_j(t)}(\varphi_j)$, when t is small enough, and the strict transforms by ρ of $f = 0$ and $g = 0$ corresponds to fibres of some tubular neighbourhoods of $U^{(j)}_{\epsilon_j(t)}(\varphi_j)$.

The plumbing graph of N is nothing but the intersection graph of the minimal good resolution ρ.

It is convenient to attach to each vertex of the plumbing graph an arrow for each branch of the strict transform of $fg = 0$ which intersects the corresponding component of the exceptional divisor. One colours the arrows corresponding to f and g differently.

This plumbing graph endowed with coloured arrows is called the *coloured plumbing graph* of the initial data (π, f, g). The triple $(\partial N, S_f, S_g)$ with that Waldhausen decomposition will be called the *coloured plumbing data* associated to the initial data (π, f, g).

THEOREM 2.16 (W. Neumann [**21**]). *The topology of* (M, K, L) *determines the coloured plumbing graph of the initial data* (π, f, g).

REMARK 2.17. (a) This theorem is one of the main results of Neumann's plumbing calculus [**21**] (see Theorem 5.6 ('Γ can be recovered from G'); the main move is 'step 1' (second half p. 323) which has to be read backwards).

In fact, Neumann gives a canonical procedure which leads from the minimal Waldhausen decomposition of (M, K, L) to the coloured plumbing graph of (π, f, g). In particular this procedure sets up a bijection between the intrinsic components of (M, K, L) and the rupture vertices of the coloured plumbing graph.

(b) Our situation is somewhat simpler for the three following reasons.

(i) The base spaces of the Seifert fibrations are all orientable. By this we avoid several problems; for instance with the manifold called Q by Waldhausen [**25**]. As a consequence, several Neumann moves are not relevant for us.

(ii) All Waldhausen ε are equal to $+1$ (so all Neumann edge signs are $+$) because our plumbings come from complex geometry. Hence one does not need plumbings more general than those considered by [**9**].

(iii) We are in a 'relative' situation, that is, we are given a non-empty link inside the 3-manifold. This greatly smoothes the discussion about lens-spaces and torus-bundles over the circle because the presence of a 'non-trivial link' (see Definition 2.10) chooses for us a Waldhausen (or Seifert) structure.

Proof of Proposition 1.2. Let Ψ be the bijection from the set J of intrinsic components of (M, K, L) to the set B of rupture vertices of the coloured plumbing graph of (π, f, g).

Now, $\mathrm{val}_{E_{\Psi(j)}}(f) = \deg(f|\gamma_j)$ by the classical topological interpretation of a divisorial valuation. The same holds with f replaced by g.

2.4. *Topological invariance*

We now prove Theorem 5 which claims that T is a topological invariant of the initial data. From §2.3, it is enough to prove that H is a topological invariant.

We already know that if (π, u, v) and (π', u', v') are topologically equivalent, then their coloured plumbing graphs are isomorphic. In particular the corresponding intersection matrices are the same. Hence what we have to prove is that the set $\{\mathrm{val}_{E_\beta}(f)\}_{\beta\in B}$ is determined by the coloured plumbing graph. To do this we use an argument which goes back to Mumford [**20**].

Let $\{E_\omega\}_{\omega\in\Omega}$ be the set of all irreducible components of the exceptional divisor of ρ. Write $m_\omega = \mathrm{val}_{E_\omega}(f)$. The divisor of $f\circ\rho$ is equal to

$$\sum_{\omega\in\Omega} m_\omega E_\omega + f'$$

where f' is the strict transform of f.

For each $\eta\in\Omega$, one has $(f\circ\rho)\cdot E_\eta = 0$. Hence the $\{m_\omega\}_{\omega\in\Omega}$ are solutions of the system of linear equations:

$$\left(\sum_{\omega\in\Omega} m_\omega E_\omega\right)\cdot E_\eta = -f'\cdot E_\eta, \quad \eta\in\Omega.$$

Now the right-hand side of this equation (for a given η) is a topological invariant. (It is equal to minus the number of arrows coloured by f which are attached to E_η.) On the left-hand side, the coefficients are determined by the plumbing graph.

The coefficient matrix $(E_\omega\cdot E_\eta)$ has a non-zero determinant because the corresponding intersection form is negative definite (see [**20**]). Hence the m_w are the unique solutions of this system of linear equations.

2.5. *The projection formula and the equality $D = C$*

We shall use the projection formula for finite morphisms and Weil divisors on surfaces (see [**7**, Appendix A, p. 426]).

As considered in the introduction, let Δ_α be a branch of Δ which is not one of the coordinate axes. Let C_λ be an irreducible component of the critical locus of π whose image by π is Δ_α. One has $(u=0)\cdot\pi_*(C_\lambda) = (u=0)\cdot\delta_\lambda\Delta_\alpha$ where δ_λ is the degree of the restriction $\pi\colon C_\lambda \longrightarrow \Delta_\alpha$. By the projection formula,

$$(u=0)\cdot\pi_*(C_\lambda) = (u\circ\pi = 0)\cdot C_\lambda = (f=0)\cdot C_\lambda.$$

Hence $(f=0)\cdot C_\lambda = \delta_\lambda(u=0)\cdot\Delta_\alpha$.

Similarly $(g=0)\cdot C_\lambda = \delta_\lambda(v=0)\cdot\Delta_\alpha$.

By taking quotients on both sides we have the equality $c_\lambda = d_\alpha$, which easily implies $C = D$.

3. *Proof of $D = T$*

3.1.

As stated in the introduction, we assume that the set $(\Delta_\alpha)_{\alpha\in A}$ of the branches of Δ at 0 which are not the coordinate axes is non-empty.

We consider the minimal embedded resolution of the curve $\{uv=0\}\cup\Delta$ in $\mathbf{C}^2$ at O. We endow the resolution tree with arrows corresponding to the branches of the strict transform of $\{uv=0\}\cup\Delta$ as we did in [**15**]. Vertices of the resolution tree which meet at least three edges or arrows are the rupture points of the tree. Again as in [**15**], each rupture point contributes to an intrinsic Seifert component of the minimal Waldhausen decomposition of $\mathbf{S}^3$ compatible with the link associated to $\{uv=0\}\cup\Delta$

at O. In that resolution tree we look at the geodesic G which goes from the arrow corresponding to the strict transform of $u=0$ to the one corresponding to the strict transform of $v=0$. As Δ is non-empty, this geodesic meets rupture vertices $p_1,\dots,p_k$, with $k \geqslant 1$, in that order. For $1 \leqslant j \leqslant k$, let $\gamma(p_j)$ be a regular leaf of the Seifert component corresponding to p_j.

Proposition 3.1. *Any regular leaf of a Seifert component of the minimal Waldhausen decomposition of* $\mathbf{S}^3$ *compatible with the link associated to* $\{uv=0\}\cup\Delta$ *at* O *and any leaf corresponding to a branch of* Δ *is satellite of some* $\gamma(p_j)$*,* $1 \leqslant j \leqslant k$*. Moreover for any* j*,* $1 \leqslant j \leqslant k$*, there is a branch of* Δ *which is satellite to* $\gamma(p_j)$*.*

Proof. This is a general fact about links associated to germs of analytic functions at the origin in $\mathbf{C}^2$ (see [**3**, §9] or [**19**, Chapter 3]).

We define for each $\gamma(p_j)$, $1 \leqslant j \leqslant k$, the ratio

$$d'_j := \frac{\mathscr{L}(A,\gamma(p_j))}{\mathscr{L}(B,\gamma(p_j))}$$

where $\mathscr{L}(A,\gamma(p_j))$ and $\mathscr{L}(B,\gamma(p_j))$ are the linking numbers of the leaves A and B defined by $u=0$ and $v=0$ respectively with $\gamma(p_j)$, in the orientation induced by the complex orientation of the interior of the ball. Let D' be the set of d'_j for $1 \leqslant j \leqslant k$. By [**19**, Chapter 3], in fact one has

$$d'_1 > \ldots > d'_k$$

and D' has exactly k elements. Proposition 3.1 implies the following corollary.

Corollary 3.2. *We have* $D = D'$.

3.2.

Since π is analytic and proper, inverse images by π of subanalytic neighbourhoods of 0 in U are subanalytic neighbourhoods of z in Z. Let us choose balls centred at 0 as subanalytic neighbourhoods of 0 in U. The boundary of the inverse image of such a ball $\mathbf{B}$ is M. The map π induces a smooth map p of (M,K,L) into $(\partial\mathbf{B},A,B)$:

$$p:(M,K,L)\longrightarrow(\partial\mathbf{B},A,B).$$

As the ramification locus of π lies inside $\Delta\cup\{uv=0\}$, the map p is a ramified cover with branch set in $\partial\mathbf{B}$ contained in $(\Delta\cap\partial\mathbf{B})\cup A\cup B$.

Proposition 3.3. *Any Waldhausen decomposition of the pair* $(\partial\mathbf{B},(\Delta\cap\partial\mathbf{B})\cup A\cup B)$ *pulls back by* p *to a Waldhausen decomposition of the pair* $(M,K\cup L)$*, such that, for each Seifert component of* $(\partial\mathbf{B},(\Delta\cap\partial\mathbf{B})\cup A\cup B)$*, its inverse image by* p *is a finite union of Seifert components of* $(M,K\cup L)$ *and images by* p *of leaves in* M *are leaves in* $\mathbf{B}$.

Corollary 3.4. *We have* $T\subset D$.

Proof. By Theorem 2.13 leaves of the minimal Waldhausen decomposition of the pair $(M,K\cup L)$ are among the leaves of the Waldhausen decomposition of $(M,K\cup L)$ obtained by Proposition 3.3 applied to the minimal decomposition of $(\partial\mathbf{B},(\Delta\cap\partial\mathbf{B})\cup A\cup B)$.

Now, let γ_ℓ be a regular leaf of a Seifert component of the minimal Waldhausen decomposition of the pair $(M, K \cup L)$. Its image $p(\gamma_\ell)$ is a leaf of the minimal decomposition of $(\partial\mathbf{B}, (\Delta \cap \partial\mathbf{B}) \cup A \cup B)$. We may assume that $p(\gamma_\ell)$ is a regular leaf. By Proposition 3.1, $p(\gamma_\ell)$ is satellite of some $\gamma(p_j)$. Now, we have

$$\deg(f \mid \gamma_\ell) = \alpha_\ell \deg(u \mid p(\gamma_\ell))$$

where α_ℓ is the degree of the map from γ_ℓ onto $p(\gamma_\ell)$ induced by p. Therefore

$$t_\ell = \frac{\alpha_\ell \deg(u \mid p(\gamma_\ell))}{\alpha_\ell \deg(v \mid p(\gamma_\ell))}$$

which shows that $T \subset D$.

3.3.

It remains to prove that $D \subset T$.

Thanks to Corollary 3.2 it is enough to prove that $D' \subset T$. To do so, we consider a rupture vertex p_j of the geodesic G as in §3.1. We prove the following proposition.

PROPOSITION 3.5. *There is a leaf μ_j of the minimal Waldhausen decomposition of the pair $(\partial\mathbf{B}, (\Delta \cap \partial\mathbf{B}) \cup A \cup B)$ which is a satellite of $\gamma(p_j)$, such that a connected component of its inverse image by p is a regular leaf v_j of the minimal Waldhausen decomposition of the pair $(M, K \cup L)$.*

This proposition implies that $D' \subset T$, because

$$\frac{\deg(f \mid v_j)}{\deg(g \mid v_j)}$$

is an element of T and

$$\frac{\deg(f \mid v_j)}{\deg(g \mid v_j)} = \frac{\alpha_j \deg(u \mid \mu_j)}{\alpha_j \deg(v \mid \mu_j)} = \frac{\deg(u \mid \mu_j)}{\deg(v \mid \mu_j)}$$

where α_j is the degree of v_j over μ_j. Now

$$\frac{\deg(u \mid \mu_j)}{\deg(v \mid \mu_j)} = \frac{\beta_j \deg(u \mid \gamma(p_j))}{\beta_j \deg(v \mid \gamma(p_j))}$$

where β_j is the winding number of μ_j around $\gamma(p_j)$. It is known that

$$\deg(u \mid \gamma(p_j)) = \mathscr{L}(A, \gamma(p_j)) \quad \text{and} \quad \deg(v \mid \gamma(p_j)) = \mathscr{L}(B, \gamma(p_j)).$$

Therefore,

$$d'_j = \frac{\mathscr{L}(A, \gamma(p_j))}{\mathscr{L}(B, \gamma(p_j))} = \frac{\deg(f \mid v_j)}{\deg(g \mid v_j)} \in T.$$

Proof of Proposition 3.5. We essentially follow the argument of [**15**]. For each $\alpha \in A$ we have the Puiseux expansion of Δ_α

$$u = c_\alpha v^{d_\alpha} + b_\alpha v^{e_\alpha} + \{\text{terms of higher degree}\}.$$

We have $D = D'$ by Corollary 3.2. We choose A_j and B_j, $1 \leqslant j \leqslant k$, such that

$$A_j < \inf\{|a_\alpha|, d_\alpha = d_j\} \leqslant \sup\{|a_\alpha|, d_\alpha = d_j\} < B_j.$$

Call $\mathscr{C}_j$ the subset of points (u, v) in U such that

$$A_j|v|^{d_j} \leqslant |u| \leqslant B_j|v|^{d_j}.$$

In a neighbourhood of O, the curves Δ_α having the same $d_\alpha = d_j$ are contained in $\mathscr{C}_j$. Now, for each α such that $d_j = d_\alpha$, we choose a rational number s_j such that

$$d_j = d_\alpha < s_j < e_\alpha.$$

Call W_j the union for α, $d_j = d_\alpha$, of the sets of points (u, v) in U, for which

$$|u - a_\alpha v^{d_\alpha}| < |v|^{s_j}.$$

In [**15**], it is proved that, when the radius of the sphere $\partial\mathbf{B}$ is small enough, the manifold $(\mathscr{C}_j - W_j) \cap \partial\mathbf{B}$ is a Seifert manifold and one of its regular leaves μ_j is a regular leaf of a Seifert component of a minimal Waldhausen decomposition of the pair

$$(\partial\mathbf{B}, (\Delta \cap \partial\mathbf{B}) \cup A \cup B).$$

Furthermore, in [**15**], it is shown that the inverse image by p of $(\mathscr{C}_j - W_j) \cap \partial\mathbf{B}$ contains a Seifert component of a minimal Waldhausen decomposition of the pair $(M, K \cup L)$, so that a connected component of the inverse image of μ_j by p is a regular leaf ν_j of a minimal Waldhausen decomposition of the pair $(M, K \cup L)$.

REMARK 3.6. The argument given above proves that the minimal resolution graph associated to the link (M, K, L) by Neumann's plumbing calculus has at least one rupture vertex when the set $(\Delta_\alpha)_{\alpha\in A}$ is non-empty. However, non-reduced links (M, K, L) are exactly the ones whose resolution graph has no rupture vertex. This proves that (M, K, L) is always reduced when $(\Delta_\alpha)_{\alpha\in A}$ is non-empty as promised in the introduction.

4. *Consequences and examples*

4.1.

Notice that, if u and v are both transverse to the branches of Δ, we have $D = \{1\}$.

4.2.

Consider the minimal embedded resolution of Δ in U. If v is transverse to the branches of Δ, this resolution is also the embedded resolution of $\{v = 0\} \cup \Delta$. Furthermore, if none of the branches of Δ is non-singular, the minimal embedded resolution of Δ in U is also the embedded resolution of $\{uv = 0\} \cup \Delta$.

In any case, following what we said in §3.1, we consider the minimal embedded resolution of $\{uv = 0\} \cup \Delta$ in U. The ratios d_α are obtained by computing the Hironaka numbers of (u, v) on the components of the exceptional divisor which correspond to the rupture vertices lying on the geodesic G which connects the strict transforms of $\{u = 0\}$ and $\{v = 0\}$.

Since Hironaka numbers are ratios of intersection numbers, one can compute discriminantal ratios from the tree of the minimal embedded resolution of $\{uv = 0\} \cup \Delta$ and the ratios of the valuations of u and v on the exceptional components of this resolution.

Since one of the two coordinate axes is certainly transverse to the branches Δ_α, it is convenient to define the contact number $\delta_O(L, X)$ of a germ of non-singular curve (L, O) and a branch of plane curve (C, O) and (C_2, O) at O (see [**8**, 1.4]) as the quotient of the intersection number of L and C at O and the multiplicity of C at O:

$$\delta_O(L, X) := \frac{(L \cdot C)_O}{m(C, O)}.$$

When v is chosen to be transverse to Δ_α at 0, the number d_α is the contact number $\delta_O(u = 0, \Delta_\alpha)$. When the branch Δ_α is singular at 0, d_α is an integer bounded by the first Puiseux pair of Δ at 0 or the first Puiseux pair, when the contact of $u = 0$ with Δ_α at 0 is maximal. When the branch Δ_α is non-singular at 0, d_α can be any positive integer.

4.3.

Let us give some examples. We consider hypersurfaces of multiplicity 2 and 3, which are cyclic ramified coverings of the complex plane. In these cases, the discriminant is given explicitly by the equation of the hypersurface. The computation of D is immediate by standard methods of plane curve singularities. Then, following Laufer in [**13**] (see also [**16**]), we can obtain a good resolution $\rho: Y \longrightarrow Z$ and compute H (see §1.3). In each case one can check that $D = H$ and hence that $D = T$.

(a) First, assume that the discriminant Δ is a branch, that is, Δ is analytically irreducible at 0. We suppose that $\{v = 0\}$ is transverse to Δ and $\{u = 0\}$ has maximal contact with Δ at 0. In this case, the minimal embedded resolution tree of $\{uv = 0\} \cup \Delta$ has only one rupture point on the geodesic which links the strict transforms of $\{u = 0\}$ and $\{v = 0\}$. Then, there is only one discriminantal ratio which is the first Puiseux pair of Δ. If $\{u = 0\}$ has not the maximal contact with Δ, the ratio is an integer.

Consider the example when $u^3 - v^8 = 0$ is an equation of Δ.

We proceed as follows. First, consider the resolution graph of the minimal embedded resolution of the plane curve singularity $u^3 - v^8 = 0$. This minimal

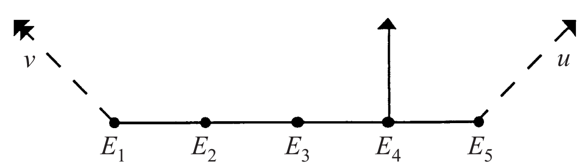

FIGURE 1.

TABLE 1.

Vertices	E_1	E_2	E_3	E_4	E_5
$E_i \cdot E_i$	-2	-3	-2	-1	-3
$v_{E_i}(u)/v_{E_i}(v)$	1	2	5/2	8/3	3

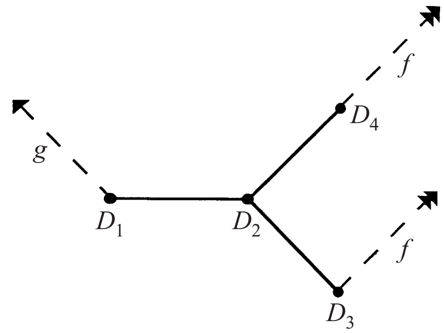

FIGURE 2.

TABLE 2.

Vertices	D_1	D_2	D_3	D_4
$D_i \cdot D_i$	-4	-1	-3	-3
$v_{D_i}(f)/v_{D_i}(g)$	2	8/3	3	3

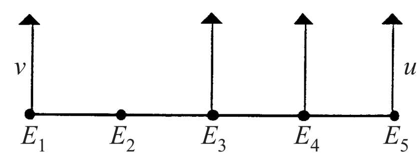

FIGURE 3.

TABLE 3.

Vertices	E_1	E_2	E_3	E_4	E_5
$E_i \cdot E_i$	-2	-3	-2	-1	-3
$v_{E_i}(u)/v_{E_i}(v)$	1	2	5/2	8/3	3

embedded resolution graph defines the minimal embedded resolution graph of $\{uv = 0\} \cup \{u^3 - v^8 = 0\}$. It suffices to add one arrow at each end of the graph (see Figure 1 and Table 1).

The minimal good resolution graph of $fg = 0$ on (Z, z) is shown in Figure 2 where we have the values given in Table 2. We see that $D = H = \{8/3\}$.

(b) Consider the case of the surface singularity (Z, z) given by

$$z^3 = (u^3 - v^8)(u^2 - v^5).$$

We consider the projection π on the (u, v)-plane. The minimal embedded resolution graph of the discriminant is shown in Figure 3 (see also Table 3).

The intersection graph of the minimal good resolution of (Z, z) is shown in Figure 4 (Table 4).

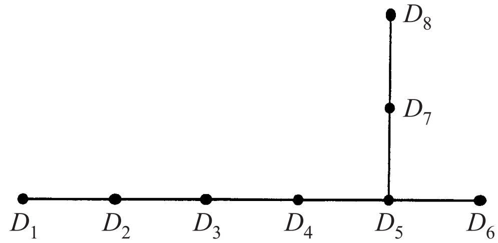

FIGURE 4.

TABLE 4.

Vertices	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
$D_i \cdot D_i$	-2	-2	-2	-2	-2	-2	-2	-2
Genus	0	0	0	0	0	1	0	0

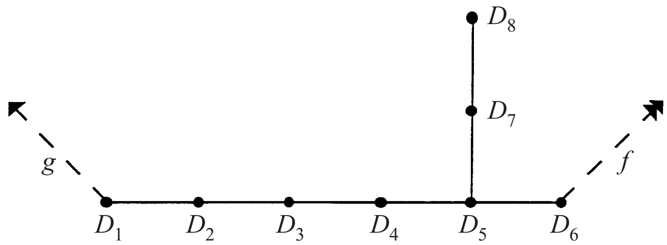

FIGURE 5.

TABLE 5.

Vertices	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
$v_{D_i}(f)/v_{D_i}(g)$	3/2	2	9/4	12/5	5/2	8/3	5/2	5/2

The intersection graph of the minimal good embedded resolution of $fg = 0$ on (Z, z), with $f := u \circ \pi$ and $g := v \circ \pi$, is shown in Figure 5 (Table 5).

In this example, we observe that D_5 and D_6 are rupture points of the minimal good resolution graph of $fg = 0$ on (Z, z), since the valence of D_5 is equal to 3 and since the vertex D_6 has genus 1.

In this case we see that $D = H = \{5/2, 8/3\}$.

4.4.

In this subsection, we give examples which illustrate how delicate our results are.

(a) Let (Z, z) be defined by

$$z^2 = (u^2 + v^5)(v^2 + u^3)$$

and let π be the projection on the (u, v)-plane. The intersection tree of the minimal embedded resolution of the discriminant is shown in Figure 6 (Table 6).

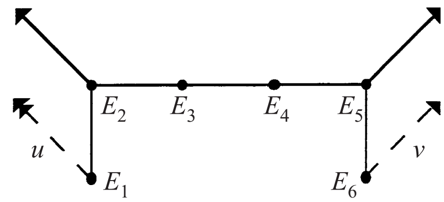

FIGURE 6.

TABLE 6.

Vertices	E_1	E_2	E_3	E_4	E_5	E_6
$E_i \cdot E_i$	-2	-1	-3	-4	-1	-2
$v_{E_i}(u)/v_{E_i}(v)$	$5/2$	$5/2$	2	1	$2/3$	$2/3$

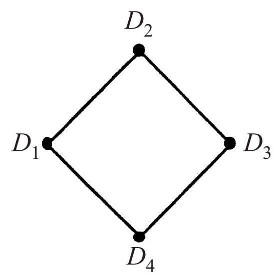

FIGURE 7.

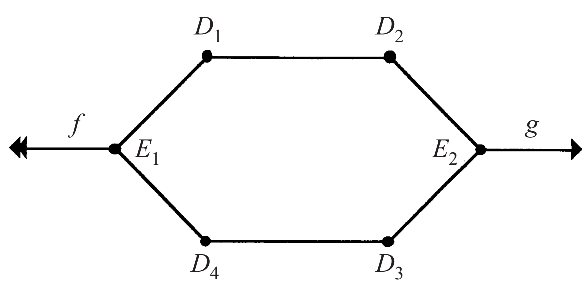

FIGURE 8.

TABLE 7.

Vertices	D_1	D_2	D_3	D_4	E_1	E_2
$D_i \cdot D_i$	-3	-4	-4	-3	-1	-1
$v_{D_i}(f)/v_{D_i}(g)$	2	1	1	2	$5/2$	$2/3$

The local link of (Z, z) is a non-reduced 3-manifold in the sense of [**25**]. The minimal resolution graph is shown in Figure 7. It has no rupture vertices. The minimal resolution graph of $fg = 0$ on (Z, z), with $f := u \circ \pi$ and $g := v \circ \pi$, is shown in Figure 8 (Table 7). The vertices E_1 and E_2 are now rupture vertices. We see that $D = H = \{5/2, 2/3\}$.

(b) Let (Z, z) be defined by

$$z^2 = y(y^2 - x^3)$$

and the projection π be defined by (f, g), where $f = u = y + x^5$ and $g = v = x$. Observe that (Z, z) is the simple singularity E_7. The minimal embedded good resolution graph of $y(y^2 - x^3) = 0$ is shown in Figure 9 (Table 8).

The intersection graph of the minimal good resolution of (Z, z) is E_7. The minimal good embedded resolution graph of $fg = 0$ on (Z, z) is shown in Figure 10 where we have the values given in Table 9.

In this last example, the intersection graph of the minimal good resolution of (Z, z) has only one rupture vertex, but two discriminantal ratios, since the other rupture vertex appears with the strict transform of $f = 0$. We have $D = H = \{3/2, 5\}$.

4.5.

The underlying theme of our paper is to determine how much information about the singularity is contained in the discriminant.

Our discriminantal ratios can be seen as 'first order' invariants of the discriminant, since they are related to first Puiseux exponents. Through Theorem 0.3 (see the introduction), they give partial information about the resolution.

However, in some cases more can be said. See, for instance, the following theorem.

THEOREM 4.1. *Suppose that the minimal embedded resolution of Δ has only one rupture point. Then the resolution graph of the minimal good resolution of the surface Z at z is starlike, that is, it is the union of k bamboos having one vertex in common* ($k \geqslant 0$).

Proof. We use the Jung resolution of Z via the projection π. Let

$$\sigma : U_1 \longrightarrow U$$

be the minimal embedded resolution of Δ and $\pi_1 : Z_1 \longrightarrow U_1$ be the pull-back of π by σ. Let $n : \bar{Z}_1 \longrightarrow Z_1$ be the normalization of Z_1. The singularities of $\bar{Z}_1$ are of Jung type, that is, they are cyclic quotients of $\mathbf{C}^2$ (see [**1**]). The inverse image by $\pi_1 \circ n$ on the set of the exceptional components corresponding to the vertices of a connected component of the complement of the rupture vertex E in the minimal embedded resolution of Δ defines distinct bamboos which only meet at the exceptional components in the inverse image by $\pi_1 \circ n$ above the rupture component of the minimal embedded resolution of Δ.

Let ρ_1 be the minimal resolution of $\bar{Z}_1$ and $\sigma_1 : Z_1 \longrightarrow Z$ be the morphism defined by the pull-back. Then, $\sigma_1 \circ n \circ \rho_1$ is a good resolution of (Z, z). Suppose that the strict transform of the rupture vertex E by $\sigma_1 \circ n$ has r components. The strict transforms by $\pi_1 \circ n \circ \rho_1$ of the exceptional components of σ define r starlike graphs which should be linked by the bamboos of the minimal resolution of the Jung singularities of $\bar{Z}_1$.

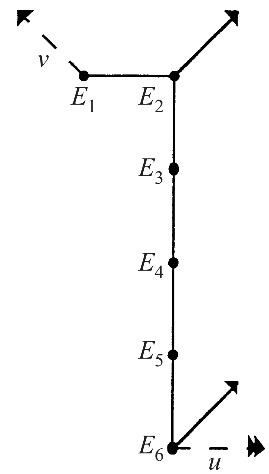

FIGURE 9.

TABLE 8.

Vertices	E_1	E_2	E_3	E_4	E_5	E_6
$E_i \cdot E_i$	-3	-1	-3	-2	-2	-1

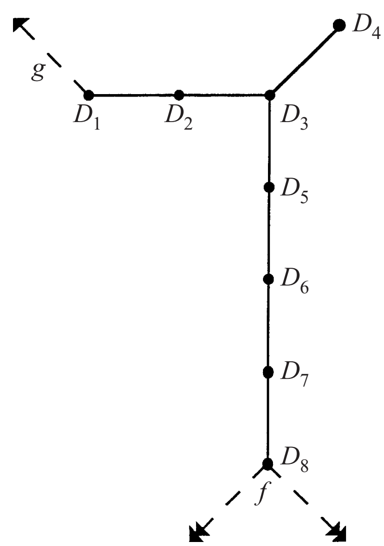

FIGURE 10.

TABLE 9.

Vertices	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
$D_i \cdot D_i$	-2	-2	-2	-2	-2	-2	-3	-1
$v_{D_i}(f)/v_{D_i}(g)$	1	4/3	3/2	3/2	5/3	2	3	5

However these bamboos meet only at one point the exceptional divisor of the resolution of $\sigma_1 \circ n \circ \rho_1$. Since the exceptional divisor of $\sigma_1 \circ n \circ \rho_1$ is connected, necessarily $r = 1$. Now, the resolution $\sigma_1 \circ n \circ \rho_1$ is not minimal in general, so that, after contractions, the graph of the minimal good resolution is starlike, but perhaps degenerate.

REMARK 4.2. Another proof of Theorem 4.1 would be to use [**21**, Theorem 5], since the fact that the minimal embedded resolution of the discriminant has only one rupture point implies that the local link of the surface singularity is a Seifert manifold.

Furthermore, notice that, when $k \geqslant 1$, only the strict transform of E by $\pi_1 \circ n \circ \rho_1$ may be of genus > 0.

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